Machine-Oriented Reasoning

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Logic

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- Logic is a formal language together with a relationship, often denoted by "$\models$", between a set of formulae and a formula:

$$\Gamma \models \varphi$$

- Formulae related by this relationship to the empty set are of special interest. If

$$\emptyset \models \varphi \text{ or } \models \varphi$$

  we say φ is valid.
- For classical logic, the deduction theorem ensures that if

$$\{\gamma_1, \ldots, \gamma_n\} \models \varphi$$

  then

$$\models (\gamma_1 \rightarrow \ldots (\gamma_{n-1} \rightarrow (\gamma_n \rightarrow \varphi)))$$

Calculus

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- A logic language together with a syntactically defined relation, often denoted by "$\vdash$", is called a calculus.

$$\Gamma \vdash \varphi$$

- Again, formulae related by this relationship to the empty set are of special interest. If

$$\emptyset \vdash \varphi \text{ or } \vdash \varphi$$

we say φ is a theorem.

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- Formally, a calculus is defined as a pair $\langle \mathcal{A}, \mathcal{R} \rangle$, where $\mathcal{A}$ is a set of formulae and $\mathcal{R}$ is a set of inference rules.
- An inference rule is usually given as:

$$\frac{\begin{array}{c}\gamma_1\\ \vdots \\ \gamma_n\end{array}}{\varphi}$$

where γ_i are the premises and φ is the conclusion. The application of the inference rule to the premises *produces* the conclusion. We also say that the conclusion is *derived* from the premises (by the inference rule).

Metatheorical Properties

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- Strong soundness:

$$\text{if } \Gamma \vdash^{C}_{\mathcal{L}} \varphi \text{ then } \Gamma \models_{\mathcal{L}} \varphi$$

- Weak soundness:

$$\text{if } \vdash^{C}_{\mathcal{L}} \varphi \text{ then } \models_{\mathcal{L}} \varphi$$

- Strong completeness:

$$\text{if } \Gamma \models_{\mathcal{L}} \varphi \text{ then } \Gamma \vdash^{C}_{\mathcal{L}} \varphi$$

- Weak completeness:

$$\text{if } \models_{\mathcal{L}} \varphi \text{ then } \vdash^{C}_{\mathcal{L}} \varphi$$

- Consistency: if there is no φ such that both $\vdash^{C}_{\mathcal{L}} \varphi$ and $\vdash^{C}_{\mathcal{L}} \neg\varphi$.

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- Because of the deduction theorem, weak properties are often enough to show a calculus has good properties. This is why automated reasoning is often referred to as *theorem proving*.
- Those good properties are not enough to ensure that a calculus is suitable for implementation.

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The set of logical symbols is given by the union of the following sets:

1. $\mathcal{P} = \{P^{n_P}, Q^{n_Q}, R^{n_R}, \ldots, P_1^{n_{P_1}}, Q_1^{n_{Q_1}}, R_1^{n_{R_1}}, \ldots\}$;
2. $\mathcal{F} = \{f^{n_f}, g^{n_g}, h^{n_h}, \ldots, f_1^{n_{f_1}}, g_1^{n_{g_1}}, h_1^{n_{h_1}}, \ldots\}$;
3. $\mathcal{C} = \{a, b, c, \ldots, a_1, b_1, c_1, \ldots\}$;
4. $\mathcal{V} = \{x, y, z, \ldots, x_1, y_1, z_1, \ldots\}$;
5. $\{\forall, \exists\}$
6. $\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\}$;
7. (,), and ,.

The set of well-formed formulae in the first-order language, denoted by $\mathsf{WFF}_{\mathcal{L}_{FO}}$, is the least set obtained by recursively applying the following:

- □ $P^n(t_1, \ldots, t_n) \in \mathsf{WFF}_{\mathcal{L}_{FO}}$, where $P^n \in \mathcal{P}$, for $t_i \in \mathcal{T}$, $0 \leq i \leq n$, $n \in \mathbb{N}$;
- □ if $\varphi, \psi \in \mathsf{WFF}_{\mathcal{L}_{FO}}$ and $x \in \mathcal{V}$, then $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$, $\forall x\varphi$ and $\exists x\varphi \in \mathsf{WFF}_{\mathcal{L}_{FO}}$.

Obs.: We'll restrict ourselves to sentences, that is, well-formed formulae with no free variables.

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An *interpretation* $\mathbb{M}$ for $(\mathcal{P}, \mathcal{F})$ consists of:

- □ a non-empty set $\mathcal{A}$ (*universe*);
- □ a function $f^{\mathbb{M}} : A^n \longrightarrow A$, for each $f^n \in \mathcal{F}$;
- □ a relation $P^{\mathbb{M}} \subseteq A^n$, for each $P^n \in \mathcal{P}$.

Let $\mathbb{M} = (\mathcal{A}, \{f^{\mathbb{M}}\}_{f \in \mathcal{F}}, \{P^{\mathbb{M}}\}_{P \in \mathcal{P}})$ be an interpretation for $(\mathcal{P}, \mathcal{F})$, $x \in \mathcal{V}$ and φ, ψ in WFF$_{\mathcal{L}_{FO}}$:

1. $\mathbb{M} \models P(t_1, \ldots, t_n)$ iff $(t_1, \ldots, t_n) \in P^{\mathbb{M}}$;
2. $\mathbb{M} \models \neg\varphi$ iff $\mathbb{M} \not\models (\varphi)$;
3. $\mathbb{M} \models \varphi \wedge \psi$ iff $\mathbb{M} \models \varphi$ e $\mathbb{M} \models \psi$;
4. $\mathbb{M} \models \varphi \vee \psi$ iff $\mathbb{M} \models \varphi$ or $\mathbb{M} \models \psi$; ;
5. $\mathbb{M} \models \varphi \rightarrow \psi$ iff $\mathbb{M} \models \neg\varphi \vee \psi$;
6. $\mathbb{M} \models \forall x \varphi$ iff $\mathbb{M} \models \varphi[a \backslash x]$ for all $a \in \mathcal{A}$;
7. $\mathbb{M} \models \exists x \varphi$ iff $\mathbb{M} \models \varphi[a \backslash x]$ for some $a \in \mathcal{A}$.

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- The development of automated reasoning tools started shorter after computers were available: the first Turing-complete programmable digital computer, the ENIAC, was operational in 1946; in 1954, a program written by Martin Davis produces a proof for the Presburger arithmetic (published in 1957).
- George Collins implemented parts of Tarski's decision procedure for elementary algebra on an IBM 704. Also published in 1957.
- Most procedures relied on (naive) enumerations of proofs and/or models.
- First results were not encouraging: even short proofs for known theorems could not be carried out by computers. In the words of Martin Davis on his program: "Its great triumph was to prove that the sum of two even numbers is even."

Presburger Procedure

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- In 1929, Presburger shows that the first order theory of addition on the natural numbers is decidable [Pre29].
- Axioms for the Presburger arithmetic are:

  1. $\neg(0 = x + 1)$
  2. $x + 1 = y + 1 \to x = y$
  3. $x + 0 = x$
  4. $x + (y + 1) = (x + y) + 1$
  5. $P(0) \wedge \forall x(P(x) \to P(x + 1)) \to \forall y P(y)$

- The procedure consists of quantifier elimination on formulae in antiprenex disjunctive normal form, whose implementation on a JOHNNIAC was reported in [Dav57].

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- The JOHNNIAC at Princeton:
  - John von Neumann Numerical Integrator and Automatic Computer, Rand Corp.
  - Vacuum tube machine.
  - 1024 40-bit word machine (yes, this is only 5KB!).
  - MFTBF (Mean Flight Time Between Failures) of 10 minutes.
  - Add time: 50ms (that's about 0.02 MIPS).
- Fischer and Rabin showed that complexity is double-exponential [FR74]. Upper bound is triple-exponential [Opp78].

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- It was devised and implemented by Newell, Shaw, and Simon [NS56, NS57, NSS57].
- It was implemented on a JOHNNIAC at the Rand Corporation, using a programming language devised by the authors.
- The calculus was based on that of the Principia: five axioms together with substitution and modus ponens. It also includes a rule for replacement of definitions. The proof is based on the construction of a tree, which handles space by using heuristics associated to detachment, backward and forward chaining. If we want to prove φ:
  - find a sentence ψ such that $\psi \rightarrow \varphi$ is provable; reduce the proof of φ to the proof of ψ (detachment);
  - if φ is of the form $\psi \rightarrow \chi$, find $\psi \rightarrow \chi'$ which is provable; reduce the proof of φ to the proof of $\chi' \rightarrow \chi$ (chaining);
  - if φ is of the form $\psi \rightarrow \chi$, find $\chi' \rightarrow \chi$ which is provable; reduce the proof of φ to the proof of $\psi \rightarrow \chi'$ (chaining).

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1.2	$(p \vee p) \to p$
1.3	$p \to (q \vee p)$
1.4	$(p \vee q) \to (q \vee p)$
1.5	$((p \vee q) \vee r) \to ((q \vee p) \vee r)$
1.6	$(p \to q) \to ((r \vee p) \to (r \vee q))$

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1.2	$(p \vee p) \rightarrow p$
1.3	$p \rightarrow (q \vee p)$
1.4	$(p \vee q) \rightarrow (q \vee p)$
1.5	$((p \vee q) \vee r) \rightarrow ((q \vee p) \vee r)$
1.6	$(p \rightarrow q) \rightarrow ((r \vee p) \rightarrow (r \vee q))$

2.01	$?(p \rightarrow \neg p) \rightarrow \neg p$

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1.2	$(p \vee p) \to p$	
1.3	$p \to (q \vee p)$	
1.4	$(p \vee q) \to (q \vee p)$	
1.5	$((p \vee q) \vee r) \to ((q \vee p) \vee r)$	
1.6	$(p \to q) \to ((r \vee p) \to (r \vee q))$	
2.01	$?(p \to \neg p) \to \neg p$	
1.2	$!(A \vee A) \to A$	$[1.2, \text{similarity}]$

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1.2	$(p \vee p) \to p$	
1.3	$p \to (q \vee p)$	
1.4	$(p \vee q) \to (q \vee p)$	
1.5	$((p \vee q) \vee r) \to ((q \vee p) \vee r)$	
1.6	$(p \to q) \to ((r \vee p) \to (r \vee q))$	
2.01	$?(p \to \neg p) \to \neg p$	
1.2	$!(A \vee A) \to A$	$[1.2, \text{similarity}]$
$1.2'$	$!(\neg B \vee \neg B) \to \neg B$	$[1.2, \text{substitution}]$

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1.2	$(p \vee p) \rightarrow p$	
1.3	$p \rightarrow (q \vee p)$	
1.4	$(p \vee q) \rightarrow (q \vee p)$	
1.5	$((p \vee q) \vee r) \rightarrow ((q \vee p) \vee r)$	
1.6	$(p \rightarrow q) \rightarrow ((r \vee p) \rightarrow (r \vee q))$	
2.01	$?(p \rightarrow \neg p) \rightarrow \neg p$	
1.2	$!(A \vee A) \rightarrow A$	[1.2, similarity]
$1.2'$	$!(\neg B \vee \neg B) \rightarrow \neg B$	[1.2, substitution]
$1.2''$	$!(B \rightarrow \neg B) \rightarrow \neg B$	[$1.2'$, replacement]

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1.2	$(p \lor p) \to p$	
1.3	$p \to (q \lor p)$	
1.4	$(p \lor q) \to (q \lor p)$	
1.5	$((p \lor q) \lor r) \to ((q \lor p) \lor r)$	
1.6	$(p \to q) \to ((r \lor p) \to (r \lor q))$	
2.01	$?(p \to \neg p) \to \neg p$	
1.2	$!(A \lor A) \to A$	[1.2, similarity]
$1.2'$	$!(\neg B \lor \neg B) \to \neg B$	[1.2, substitution]
$1.2''$	$!(B \to \neg B) \to \neg B$	[$1.2'$, replacement]
$1.2'''$	$!(p \to \neg p) \to \neg p$	[$1.2''$, replacement]

Example - Detachment Method

$2.01 \quad ?(p \to q) \to ((q \to r) \to (p \to r))$

Example - Detachment Method

2.01 $?(p \to q) \to ((q \to r) \to (p \to r))$

2.04 $!(A \to (B \to C)) \to (B \to (A \to C))$ [theorem]

Example - Detachment Method

2.01 $?(p \to q) \to ((q \to r) \to (p \to r))$

2.04 $!(A \to (B \to C)) \to (B \to (A \to C))$ [theorem]

2.04′ $!((q \to r) \to ((p \to q) \to (p \to r))) \to ((p \to q) \to ((q \to r) \to (p \to r)))$ [2.04, substitution]

Example - Detachment Method

$$2.01 \quad ?(p \to q) \to ((q \to r) \to (p \to r))$$

$$2.04 \quad !(A \to (B \to C)) \to (B \to (A \to C)) \qquad \text{[theorem]}$$

$$2.04' \quad !((q \to r) \to ((p \to q) \to (p \to r))) \to ((p \to q) \to ((q \to r) \to (p \to r)))$$

$$[2.04, \text{substitution}]$$

$$2.04'L \quad ?(q \to r) \to ((p \to q) \to (p \to r))$$

Example - Detachment Method

2.01	$?(p \to q) \to ((q \to r) \to (p \to r))$	
2.04	$!(A \to (B \to C)) \to (B \to (A \to C))$	[theorem]
$2.04'$	$!((q \to r) \to ((p \to q) \to (p \to r))) \to ((p \to q) \to ((q \to r) \to (p \to r)))$	[2.04, substitution]
$2.04'L$	$?(q \to r) \to ((p \to q) \to (p \to r))$	
2.05	$!(A \to B) \to ((C \to A) \to (C \to B))$	[theorem]

Example - Detachment Method

2.01	$?(p \to q) \to ((q \to r) \to (p \to r))$	
2.04	$!(A \to (B \to C)) \to (B \to (A \to C))$	[theorem]
$2.04'$	$!((q \to r) \to ((p \to q) \to (p \to r))) \to ((p \to q) \to ((q \to r) \to (p \to r)))$	[2.04, substitution]
$2.04'L$	$?(q \to r) \to ((p \to q) \to (p \to r))$	
2.05	$!(A \to B) \to ((C \to A) \to (C \to B))$	[theorem]
$2.05'$	$!(q \to r) \to ((p \to q) \to (p \to r))$	[2.05, substitution]

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$$
\begin{array}{rll}
2.01 & ?(p \to q) \to ((q \to r) \to (p \to r)) & \\
 & & \\
2.04 & !(A \to (B \to C)) \to (B \to (A \to C)) & [\textsf{theorem}] \\
2.04' & !((q \to r) \to ((p \to q) \to (p \to r))) \to ((p \to q) \to ((q \to r) \to (p \to r))) & \\
 & & [2.04, \textsf{substitution}] \\
2.04'L & ?(q \to r) \to ((p \to q) \to (p \to r)) & \\
2.05 & !(A \to B) \to ((C \to A) \to (C \to B)) & [\textsf{theorem}] \\
2.05' & !(q \to r) \to ((p \to q) \to (p \to r)) & \\
 & & [2.05, \textsf{substitution}] \\
2.01 & !(p \to q) \to ((q \to r) \to (p \to r)) & \\
 & & [2.05', 2.04', \textsf{detachment}]
\end{array}
$$

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2.01	$?(p \to q) \to ((q \to r) \to (p \to r))$	
2.04	$!(A \to (B \to C)) \to (B \to (A \to C))$	[theorem]
$2.04'$	$!((q \to r) \to ((p \to q) \to (p \to r))) \to ((p \to q) \to ((q \to r) \to (p \to r)))$	[2.04, substitution]
$2.04'L$	$?(q \to r) \to ((p \to q) \to (p \to r))$	
2.05	$!(A \to B) \to ((C \to A) \to (C \to B))$	[theorem]
$2.05'$	$!(q \to r) \to ((p \to q) \to (p \to r))$	[2.05, substitution]
2.01	$!(p \to q) \to ((q \to r) \to (p \to r))$	[$2.05'$, $2.04'$, detachment]

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To prove $A \rightarrow C$, find B such that:

$$((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$$

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To prove $A \rightarrow C$, find B such that:

$$((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$$

$$1.2 \quad (p \vee p) \rightarrow p$$

$$2.08 \quad ?p \rightarrow p$$

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To prove $A \rightarrow C$, find B such that:

$$((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$$

$$1.2 \quad (p \vee p) \rightarrow p$$

$$2.08 \quad ?p \rightarrow p$$

$$2.07 \quad !A \rightarrow (A \vee A) \quad \text{[theorem]}$$

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To prove $A \to C$, find B such that:

$$((A \to B) \land (B \to C)) \to (A \to C)$$

$$1.2 \quad (p \lor p) \to p$$

$$2.08 \quad ?p \to p$$

$$2.07 \quad !A \to (A \lor A) \quad [\text{theorem}]$$
$$2.07' \quad !p \to (p \lor p) \quad [2.07, \text{substitution}]$$

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To prove $A \rightarrow C$, find B such that:

$$((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$$

$$1.2 \quad (p \vee p) \rightarrow p$$

$$2.08 \quad ?p \rightarrow p$$

2.07	$!A \rightarrow (A \vee A)$	[theorem]
$2.07'$	$!p \rightarrow (p \vee p)$	$[2.07,$ substitution$]$
$2.07''$	$!p \rightarrow p$	$[2.07', 1.2,$ chaining$]$

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- The British Museum algorithm enumerates all theorems in a forward manner from axioms to the proof (if there is one).
- The Logic Theory Machine works in a backward manner in order to find sub-problems from the desired axioms or known theorems.

> (...) working backward may be analogous to the ease with which a needle can find its way out of a haystack, compared with the difficulty of someone finding the lone needle in the haystack. [NSS57]

- The Logic Theory Machine could prove 38 of the first 52 theorems on Chapter 2 of the Principia (the remainder 14 problems could not be solved because of memory limitations).
- The program could use results previously proved: there was *learning* in theorem proving already!
- Backward reasoning is still one of the most used techniques in automated reasoning.

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- The JOHNNIAC at Rand: by 1956, tubes had already been replaced by transistors, main memory had been expanded to 4096 words and a drum, the external storage device, of 12K had been added; MFTBF was around 100 hours.
- It took ten seconds to prove:

$$(2.01) \qquad (p \rightarrow \neg p) \rightarrow \neg p.$$

- It took 12 minutes, using all 38 theorems already proved, to prove

$$(2.45) \qquad \neg(p \vee q) \rightarrow \neg p.$$

- Failure to produce a proof was reported after 23 minutes for:

$$(2.31) \qquad p \vee (q \vee r) \rightarrow (p \vee q) \vee r.$$

- The Logic Theory Machine is not a complete procedure (because of the way it deals with substitution and how it uses heuristics to guide the search for a proof).

Wang and Gao

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- It must be emphasised that the Logic Theory Machine was not designed for efficiency: it tries to mimic the way humans produce proofs.
- A calculus designed for efficiency was that of Wang: it's a two-sided sequent system, whose inference rules remove one operator each time.
- In the case of a split, both sequents must be proved.
- This work started at IBM in the Summer of 1958 and then moved to the Bell Labs in 1959-1960 [Wan60a, Wan60b, Wan61].
- Three programs were developed:
  - One for propositional logic;
  - One for the decidable part of predicate logic;
  - One for all predicate logic.

Example

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Resolution

$p, \neg q \wedge r$	$\Rightarrow$	$p \wedge r$	[Initial Sequent]
$p, \neg q, r$	$\Rightarrow$	$p \wedge q$	[Left conjunction removal]
p, r	$\Rightarrow$	$q, p \wedge r$	[Left negation removal]

$p, r \quad \rightarrow \quad q, p$
[Right conjunction removal]

$p, r \quad \rightarrow \quad q, r$
[Right conjunction removal]

Success?

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- The first program, the one dedicated to propositional logic, took 37 minutes to prove the 220 theorems from the Principia.
- Techniques, based on pattern matching, were used during enumeration of substitutions (somewhat anticipating the results from Prawitz).

It is also worth noting that other authors have responded heavily to the heuristics proposed by Newell, Shaw, and Simon. See, for instance,

- B. Dunham and R. Fridshal and G. L. Sward: "A non-heuristic program for proving elementary logical theorems" [DFS59].
- The procedure used rules as splitting and pure literal reduction, which is later used by Davis and Putnam in their satisfiability procedure.

published under the chapter "Pattern recognition and machine learning" in the First International Conference on Information Processing, in Paris.

Paul Gilmore and Herbert Gelernter

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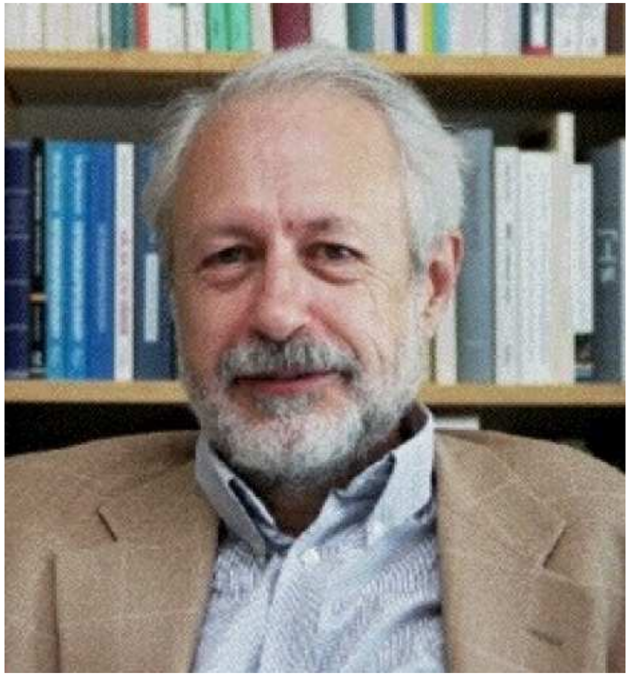
Resolution

- IBM Research at the Lamb Estate in Croton-on-Hudson, NY.
- Implemented Beth's semantic tableaux for first-order logic (although it was actually closer to Hintikka's method) in 1958 [Gel59, Gil60, Gil70].
- They lately got to know – and acknowledge the fact – that theirs were not the first implementation of a procedure for first-order logic (IBM 704).
  - Take as input a negated first-order formula in prenex form with matrix in disjunctive normal form.
  - Search for constants is about the same that the one proposed by Prawitz.

Dag Prawitz



- Prawitz developed a general calculus for predicate logic based on contradiction.
  - A derivation is a sequence of pairs: $(\phi, F), \ldots$
  - Each pair in the sequence is derived from the previous by application of "usual" inference rules for formulae without quantifiers.
  - A derivation from a quantified formula relies on some enumeration of the constants.
  - If the derivation contains two pairs (ψ, T) and (ψ, F), then a contradiction is found. Thus, the original formula is valid.

Example: $\forall x P(x) \rightarrow \exists y P(y)$

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Resolution

□ $(\forall x P(x) \rightarrow \exists y P(y), F)$

Example: $\forall x P(x) \to \exists y P(y)$

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- $(\forall x P(x) \to \exists y P(y), F)$
- $(\forall x P(x) \to \exists y P(y), F), (\forall x P(x), T), (\exists y P(y), F)$

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Resolution

- $(\forall x P(x) \to \exists y P(y), F)$
- $(\forall x P(x) \to \exists y P(y), F), (\forall x P(x), T), (\exists y P(y), F)$
- By the semantics of the universal operator, then for any constant c, we have that $(P(c), T)$. We consider an enumeration of all constants, $c_1, c_2, \ldots$, and take any constant:

Example: $\forall x P(x) \rightarrow \exists y P(y)$

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▷ Example: $\forall x P(x) \rightarrow \exists y P(y)$

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- $(\forall x P(x) \rightarrow \exists y P(y), F)$
- $(\forall x P(x) \rightarrow \exists y P(y), F), (\forall x P(x), T), (\exists y P(y), F)$
- By the semantics of the universal operator, then for any constant c, we have that $(P(c), T)$. We consider an enumeration of all constants, $c_1, c_2, \ldots$, and take any constant:
- $(P(c_1), T)$

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- $(\forall x P(x) \rightarrow \exists y P(y), F)$
- $(\forall x P(x) \rightarrow \exists y P(y), F), (\forall x P(x), T), (\exists y P(y), F)$
- By the semantics of the universal operator, then for any constant c, we have that $(P(c), T)$. We consider an enumeration of all constants, $c_1, c_2, \ldots$, and take any constant:
- $(P(c_1), T)$
- By the semantics of the existential operator, $(\exists y P(y), F)$ implies that for any constant c, we have that $P(c)$ is false. We select a constant that has been used before:

Example: $\forall x P(x) \to \exists y P(y)$

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- $(\forall x P(x) \to \exists y P(y), F)$
- $(\forall x P(x) \to \exists y P(y), F), (\forall x P(x), T), (\exists y P(y), F)$
- By the semantics of the universal operator, then for any constant c, we have that $(P(c), T)$. We consider an enumeration of all constants, $c_1, c_2, \ldots$, and take any constant:
- $(P(c_1), T)$
- By the semantics of the existential operator, $(\exists y P(y), F)$ implies that for any constant c, we have that $P(c)$ is false. We select a constant that has been used before:
- $(P(c_1), F)$

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Some notes

- This is a tableau!
- Constant selection is slightly more complicated than what was outlined in the example.
- Prawitz coded the procedure in a programming language he devised.
- His father, Håkan Prawtiz, hand-translated the code into machine code in 1957: FACIT EDB (transistorised computer), 2049 40-bits machine words, 8192 words drum.
- In 1958, Neri Voghera tested the procedure in a number of examples.
  - Outlined at the first IFIP, in Paris, 1960.
  - Published in 1960 [PPV60].
- The search for constants in Prawitz procedure were far from optimal.
- Prawitz introduced the use of *unification* in 1960: the method relied on the use of meta-variables which were replaced when *needed*, instead of using a fixed sequence.

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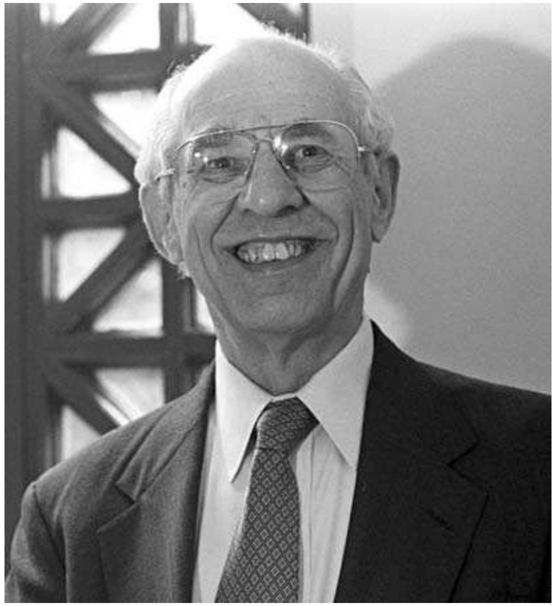
None of the mentioned procedures allowed functional symbols. Davis-Putnam, 1960, introduces Skolem functions and the Herbrand universe into the world of automated deduction. It also introduces clausal form. Finally, they proposed the unit resolution rule (which is there called "rule for the elimination of one-literal clauses") [DP60, DLL62].



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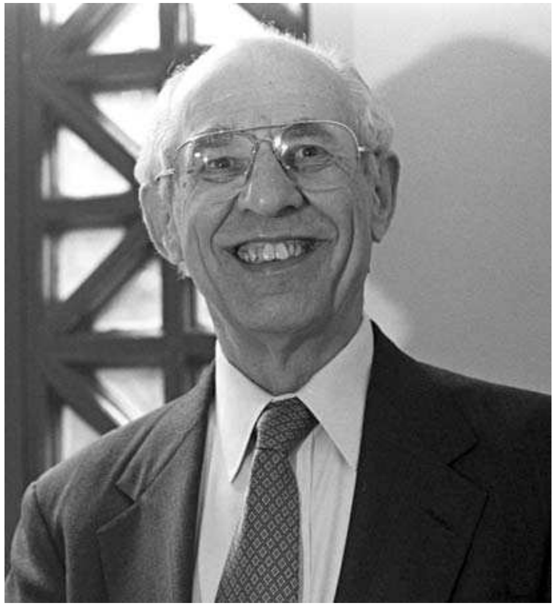

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$$\varphi \stackrel{\text{def}}{=} Q_1 x_1 \ldots Q_n x_n \psi$$

where

- $\longrightarrow$ φ is a sentence: no free variables.
- $\longrightarrow$ Each Q_i, $1 \leq i \leq n$, is a quantifier.
- $\longrightarrow$ ψ is in Negation Normal Form (NNF).
- $\longrightarrow$ ψ is the matrix of φ.
- $\longrightarrow$ x_i, $1 \leq i \leq n$, is free in ψ.

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Let

$$\varphi \stackrel{\text{def}}{=} Q_1 x_1 \ldots Q_n x_n \psi$$

be a formula in Prenex Normal Form where Q_i is the leftmost existential quantifier. If φ is satisfiable, then

$$\varphi' \stackrel{\text{def}}{=} Q_1 x_1 \ldots Q_{i-1} x_{i-1} Q_{i+1} x_{i+1} \ldots Q_n x_n \psi[x_i \mapsto f^{i-1}(x_1, \ldots, x_{i-1})]$$

where f^{i-1} is a new functional symbol, is satisfiable.

Obs.: The transformation is *satisfiability preserving*: if there is a model $\mathbb{M}$ for φ, then there is a model $\mathbb{M}$' for φ'.
Note that this is enough for proof methods based on contradiction: if there is no model $\mathbb{M}$' for φ', then there is no model $\mathbb{M}$ for φ.

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$$\neg(\forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x)))$$

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$$\neg(\forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x)))$$
$$\neg(\forall x(P(x) \to Q(x)) \to (\forall y P(y) \to \forall z Q(z)))$$

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$$\neg(\forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x)))$$
$$\neg(\forall x(P(x) \to Q(x)) \to (\forall y P(y) \to \forall z Q(z)))$$
$$\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \neg \forall z Q(z)$$

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$$\neg(\forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x)))$$
$$\neg(\forall x(P(x) \to Q(x)) \to (\forall y P(y) \to \forall z Q(z)))$$
$$\forall x(\neg P(x) \lor Q(x)) \land \forall y P(y) \land \neg\forall z Q(z)$$
$$\forall x(\neg P(x) \lor Q(x)) \land \forall y P(y) \land \exists z \neg Q(z)$$

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$$\neg(\forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x)))$$
$$\neg(\forall x(P(x) \to Q(x)) \to (\forall y P(y) \to \forall z Q(z)))$$
$$\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \neg\forall z Q(z)$$
$$\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \exists z \neg Q(z)$$
$$\forall x \forall y \exists z((\neg P(x) \vee Q(x)) \wedge P(y) \wedge \neg Q(z))$$

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$$\neg(\forall x(P(x) \rightarrow Q(x)) \rightarrow (\forall x P(x) \rightarrow \forall x Q(x)))$$
$$\neg(\forall x(P(x) \rightarrow Q(x)) \rightarrow (\forall y P(y) \rightarrow \forall z Q(z)))$$
$$\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \neg\forall z Q(z)$$
$$\forall x(\neg P(x) \vee Q(x)) \wedge \forall y P(y) \wedge \exists z \neg Q(z)$$
$$\forall x \forall y \exists z((\neg P(x) \vee Q(x)) \wedge P(y) \wedge \neg Q(z))$$
$$\forall x \forall y((\neg P(x) \vee Q(x)) \wedge P(y) \wedge \neg Q(f^2(x, y)))$$

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Let $\mathcal{S}$ be a set of clauses. The Herbrand Universe $\mathcal{H}$ associated with $\mathcal{S}$ is a set of ground terms constructed as follows:

- if there is f^0 which occurs in $\mathcal{S}$, then $f^0 \in \mathcal{H}$; otherwise, let $\mathcal{H} = \{a\}$;
- if $t_1, \ldots, t_n$ are terms in $\mathcal{H}$ and f^n occurs in $\mathcal{S}$, then $f^n(t_1, \ldots, t_n) \in \mathcal{H}$.

That is, the Herbrand Universe (for a set) is the set of all ground terms (based on that set).

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- No functions: $\mathcal{S} = \{P(x) \vee Q(x), P(x), \neg Q(x)\}$

$$\mathcal{H} = \{a\}$$

- One function: $\mathcal{S} = \{P(x) \vee Q(x), P(x), \neg Q(f(x))\}$

$$\mathcal{H} = \{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$$

- One function (our working example):
$\mathcal{S} = \{P(x) \vee Q(x), P(y), \neg Q(f(x,y))\}$

$$\mathcal{H} = \{a, f(a,a), f(a, f(a,a)), f(f(a,a),a), f(f(a,a), f(a,a)), \ldots\}$$

- Two functions: $\mathcal{S} = \{P(x) \vee Q(x), P(f(x)), \neg Q(g(x))\}$

$$\mathcal{H} = \{a, f(a), g(a), f(f(a)), f(g(a)), g(f(a)), g(g(a)), \ldots\}$$

Herbrand Base

Let $\mathcal{S}$ be a set of clauses and $\mathcal{H}$ its Herbrand universe. The Herbrand base $\mathcal{B}$ for $\mathcal{S}$ is the set of all ground atoms $P^n(t_1, \ldots, t_n)$, where P^n occurs in $\mathcal{S}$ and t_i is in $\mathcal{H}$, for $i = 1, \ldots, n$.
Examples:

- No functions: $\mathcal{S} = \{P(x) \vee Q(x), P(x), \neg Q(x)\}, \mathcal{H} = \{a\}$

$$\{P(a), Q(a)\}$$

- One function:
  $\mathcal{S} = \{P(x) \vee Q(x), P(x), \neg Q(f(x))\}, \mathcal{H} = \{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$

$$\{P(a), Q(a), P(f(a)), Q(f(a)), P(f(f(a))), Q(f(f(a))), \ldots\}$$

- For our working example: $\mathcal{S} = \{\neg P(x) \vee Q(x), P(y), \neg Q(f^2(x, y))\}$,
  $\mathcal{H} = \{a, f^2(a, a), f^2(a, f^2(a, a)), f^2(f^2(a, a), a), f^2(f^2(a, a), f^2(a, a)), \ldots\}$

$$\{P(a), Q(a), P(f^2(a, a)), Q(f^2(a, a)), P(f^2(a, f(a, a))), P(f^2(f(a, a), a)), P(f^2(f(a, a), f^2(a, a)))$$

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Let $\mathcal{S}$ be a set of clauses and $\mathcal{H}$ be its Herbrand universe.

- A *Herbrand interpretation* provides:
  - A domain: $\mathcal{H}$
  - An assignment for constants to an element of the domain;
  - An assignment for functional symbols to an element of the domain;
  - An assignment for predicate symbols to $\{\bot, \top\}$

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Let $\mathcal{S}$ be a set of clauses, $\mathcal{H}$ be its Herbrand universe and $\mathcal{B}$ be a Herbrand base for $\mathcal{S}$. Let $\mathcal{I} = \langle \mathcal{H}, \mathcal{B} \rangle$. We say that $\mathcal{I}$ is a *Herbrand interpretation* if $t^{\mathcal{B}} = t$ for all terms t.
That is,

- If c is a constant, then $c^{\mathcal{I}} = c$;
- If f is a n-ary functional symbol, then $f^{\mathcal{I}}(h_1, \ldots, h_n) = f(h_1, \ldots, h_n)$.
- There are no restrictions for predicate symbols.
  If $\mathcal{B} = \{P_1, \ldots, P_n, \ldots\}$, then take $\mathcal{I} = \{m_1, \ldots, m_n, \ldots\}$, where $m_i = P_i$ or $m_i = \neg P_i$.

Example

$$\mathcal{S} = \{P(x) \vee Q(x), P(y), \neg Q(f^2(x,y))\}$$

$$\mathcal{H} = \{a, f^2(a,a), f^2(a, f^2(a,a)), f^2(f^2(a,a),a), f^2(f^2(a,a), f^2(a,a)), \ldots\}$$

$$\{P(a), Q(a), P(f^2(a,a)), Q(f^2(a,a)), P(f^2(a,f(a))), P(f^2(f(a),a)), P(f^2(f(a), f^2(a,a))), \ldots$$

Possible interpretations:

$\mathcal{I}_1 = \{P(a), Q(a), P(f^2(a,a)), Q(f^2(a,a)), P(f^2(a, f^2(a,a))), P(f^2(f^2(a,a),a)), \ldots\}$

$\mathcal{I}_2 = \{\neg P(a), Q(a), P(f^2(a,a)), Q(f^2(a,a)), P(f^2(a, f^2(a,a))), P(f^2(f^2(a,a),a)), \ldots\}$

$\mathcal{I}_3 = \{P(a), \neg Q(a), P(f^2(a,a)), Q(f^2(a,a)), P(f^2(a, f^2(a,a))), P(f^2(f^2(a,a),a)), \ldots\}$

$\ldots$

Ground Instances of Clauses

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Let $\mathcal{S}$ be a set of clauses and $\mathcal{H}$ its Herbrand universe. A ground instance of a clause C, $C \in \mathcal{S}$, for a Herbrand Base $\mathcal{H}$ (of $\mathcal{S}$) is the clause $C' = C[t_{ji} \backslash x_i]$, for all x_i occurring in C and $t_{ji} \in \mathcal{H}$.
Examples:

- No functions: $\mathcal{S} = \{P(x) \vee Q(x), P(x), \neg Q(x)\}, \mathcal{H} = \{a\}$, $\mathcal{B} = \{P(a), Q(a)\}$
  - $P(a) \vee Q(a), P(a), \neg Q(a)$ are the only ground instances for $\mathcal{B}$
- For our working example: $\mathcal{S} = \{\neg P(x) \vee Q(x), P(y), \neg Q(f^2(x,y))\}$, $\mathcal{H} =$ $\{a, f^2(a,a), f^2(a, f^2(a,a)), f^2(f^2(a,a),a), f^2(f^2(a,a), f^2(a,a)), \ldots\}$, $\mathcal{B} = \{\{P(a), Q(a), P(f^2(a,a)), Q(f^2(a,a)), \ldots\}$
  - $P(a) \vee Q(a)$
  - $P(f^2(a,a)) \vee Q(f^2(a,a))$
  - $P(f^2(a, f^2(a,a))) \vee Q(f^2(a, f^2(a,a)))$
  - $\ldots$

Satisfiability of Ground Clauses

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Let $\mathcal{S}$ be a set of clauses and a $\mathcal{I}$ be a Herbrand interpretation for $\mathcal{S}$.

- Let C' be a ground instance of a clause C in $\mathcal{S}$. We say that $\mathcal{I}$ satisfies C', denoted by $\mathcal{I} \models C'$, if $C' \cap \mathcal{I} \neq \emptyset$ (note that clauses are regarded as set of literals).
- We say that $\mathcal{I}$ satisfies C in $\mathcal{S}$ if, and only if, for all ground clauses C' of C, we have that $\mathcal{I} \models C'$.

Example: $\mathcal{S} = \{\neg P(x) \vee Q(f(x))\}$

$\mathcal{I} = \{P(a), \neg Q(a), P(f(a)), Q(f(a)), \neg Q(f(f(a))), \ldots\}$

$\mathcal{I} \models \neg P(a) \vee Q(f(a))$
$\mathcal{I} \not\models \neg P(f(a)) \vee Q(f(f(a)))$

Unsatisfiability

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Let $\mathcal{S}$ be a set of clauses. $\mathcal{S}$ is unsatisfiable if and only if for every Herbrand interpretation $\mathcal{I}$ for $\mathcal{S}$ there is a ground clause C' of C in $\mathcal{S}$ such that $\mathcal{I} \not\models C'$.

Example: $\{\neg P(x), P(a)\}$

There are only two possible Herbrand interpretations:

- $\mathcal{I}_1 = \{P(a)\}$: does not satisfy the first clause;
- $\mathcal{I}_2 = \{\neg P(a)\}$: does not satisfy the second clause.

Saturation

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Let $\mathcal{S}$ be a set of clauses. Let $\mathcal{T}$ be a set of terms. By $\mathcal{T}(\mathcal{S})$, we denote the saturation of $\mathcal{S}$ over $\mathcal{T}$, which is the set of all ground clauses obtainable from $\mathcal{S}$ by (uniformly) replacing variables by members of $\mathcal{T}$.

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Let $\mathcal{S}$ be a set of clauses. Let $\mathcal{T}$ be a set of terms. By $\mathcal{T}(\mathcal{S})$, we denote the saturation of $\mathcal{S}$ over $\mathcal{T}$, which is the set of all ground clauses obtainable from $\mathcal{S}$ by (uniformly) replacing variables by members of $\mathcal{T}$.

Example: $\mathcal{S} = \{\neg P(x) \vee Q(x), P(y), \neg Q(f^2(x,y))\}$

$$\mathcal{H} = \{a, f^2(a,a), f^2(a, f^2(a,a)), f^2(f^2(a,a),a), f^2(f^2(a,a), f^2(a,a)), \ldots\}$$

$$\begin{aligned}\mathcal{H}(\mathcal{S}) = \{ \quad & \neg P(a) \vee Q(a), P(a), \neg Q(f^2(a,a)),\\ & \neg P(a) \vee Q(a), P(f^2(a,a)), \neg Q(f^2(a, f^2(a,a))),\\ & \neg P(f^2(a,a)) \vee Q(f^2(a,a)), P(a), \neg Q(f^2(f^2(a,a),a)),\\ & \neg P(f^2(a,a)) \vee Q(f^2(a,a)), P(f^2(a,a)), \neg Q(f^2(f^2(a,a), f^2(a,a))),\\ & \ldots\}\end{aligned}$$

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Let $\mathcal{S}$ be a set of clauses. Let $\mathcal{T}$ be a set of terms. By $\mathcal{T}(\mathcal{S})$, we denote the saturation of $\mathcal{S}$ over $\mathcal{T}$, which is the set of all ground clauses obtainable from $\mathcal{S}$ by (uniformly) replacing variables by members of $\mathcal{T}$.

Example: $\mathcal{S} = \{\neg P(x) \vee Q(x), P(y), \neg Q(f^2(x, y))\}$

$$\mathcal{H} = \{a, f^2(a,a), f^2(a, f^2(a,a)), f^2(f^2(a,a), a), f^2(f^2(a,a), f^2(a,a)), \ldots\}$$

$$\begin{aligned} \mathcal{H}(\mathcal{S}) = \{ \quad & \neg P(a) \vee Q(a), P(a), \neg Q(f^2(a,a)), \\ & \neg P(a) \vee Q(a), P(f^2(a,a)), \neg Q(f^2(a, f^2(a,a))), \\ & \neg P(f^2(a,a)) \vee Q(f^2(a,a)), P(a), \neg Q(f^2(f^2(a,a), a)), \\ & \neg P(f^2(a,a)) \vee Q(f^2(a,a)), P(f^2(a,a)), \neg Q(f^2(f^2(a,a), f^2(a,a))), \\ & \ldots\} \end{aligned}$$

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- A set of ground literals which does not include a complementary pair is called a *model* (or interpretation).
- If $\mathcal{I}$ is a model and $\mathcal{S}$ is a set of ground clauses, then $\mathcal{I}$ is a model of $\mathcal{S}$ if, for all clauses C in $\mathcal{S}$, C contains a member of $\mathcal{I}$.
- In general, if $\mathcal{S}$ is any set of clauses and $\mathcal{H}$ is the Herbrand universe of $\mathcal{S}$, then $\mathcal{I}$ is a model of $\mathcal{S}$ just in case $\mathcal{I}$ is a model of $\mathcal{H}(\mathcal{S})$.

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If $\mathcal{S}$ is any finite set of clauses and $\mathcal{H}$ is its Herbrand Universe, then $\mathcal{S}$ is unsatisfiable if and only if some finite subset of $\mathcal{H}(\mathcal{S})$ is unsatisfiable.

Note that this theorem suggests the following refutation procedure:

1. Select $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots$ as finite subsets of the Herbrand universe $\mathcal{H}$ of $\mathcal{S}$, such that $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$, for all $i \geq 0$ and such that $\bigcup_{i=0}^{\infty} \mathcal{P}_i = \mathcal{H}$.
2. Check $\mathcal{P}_0(\mathcal{S}), \mathcal{P}_1(\mathcal{S}), \mathcal{P}_2(\mathcal{S}), \ldots$ for satisfiability.
3. By Herbrand's theorem, for some i, $\mathcal{P}_i(\mathcal{S})$ is unsatisfiable.

Level saturation procedures take $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$ to be the finite sets over which the set of clauses is saturated, where each $\mathcal{H}_{i+1}$ contains all the terms in $\mathcal{H}_i$ or whose arguments are in $\mathcal{H}_i$.
Finding *proof sets*, i.e. finite unsatisfiable subsets which are minimal (in the sense that every proper subset is satisfiable) requires guidance (heuristics).

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In order to prove φ

- Take the Skolemised prenex normal form of φ, where the matrix is in Conjunctive Normal Form, obtaining φ'
- **Repeat**
  Generate the Herbrand sets by level and check for unsatisfiability
  until a contradiction has been found.

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$$\forall x_1 \exists x_2 \forall x_3 R(x_1, x_2, x_3)$$

$$\forall x_1 \forall x_3 R(x_1, f(x_1), x_3)$$

$$R(a, f(a), a)$$

$$R(a, f(a), f(a))$$

$$R(f(a), f(f(a)), a)$$

$$R(f(a), f(f(a)), f(a))$$

$$\vdots$$

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1. Unit resolution (which is there called "Rule for the Elimination of One-Literal Clauses"):
   (a) if p, $\neg p$ are in $\mathcal{S}$, $\mathcal{S} = \{\bot\}$.
   (b) if p in $\mathcal{S}$, then
   $\mathcal{S} = \mathcal{S} \setminus \{C \mid p \in C\} \setminus \{D \mid \neg p \in D\} \cup \{D \mid D \cup \{\neg p\} \in \mathcal{S}\}$.
   (c) $\neg p$ in $\mathcal{S}$, then
   $\mathcal{S} = \mathcal{S} \setminus \{C \mid \neg p \in C\} \setminus \{D \mid p \in D\} \cup \{D \mid D \cup \{p\} \in \mathcal{S}\}$.
2. Pure Literal Elimination (which is there called "Affirmative-Negative Rule"): if l occurs only positively or negatively, then $\mathcal{S} = \mathcal{S} \setminus \{D \mid l \in D\}$.
3. Rule for Eliminating Atomic Formulas: if we can obtain (by factoring) $\mathcal{S} = \{(\varphi \vee p), (\psi \vee \neg p), R\}$, where R is free of p, then $\mathcal{S} = \{(\varphi \vee \psi), R\}$

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$$p \vee q \vee \neg r, p \vee \neg q, \neg p, r$$

$p \vee q \vee \neg r, p \vee \neg q, \neg p, r$

$p \vee q, p \vee \neg q, \neg p$

$q, \neg q$

$\bot$

$$p \vee q, \neg q, \neg p \vee q \vee \neg r$$

$p \vee q, \neg q, \neg p \vee q \vee \neg r$

$p \vee q, \neg q$

$p \vee q, \neg q$

p

$\emptyset$

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$$p \vee \neg q, \neg p \vee q, q \vee \neg r, \neg q \vee \neg r$$

$p \vee \neg q, \neg p \vee q, q \vee {\color{red}\neg r}, \neg q \vee {\color{red}\neg r}$

$p \vee \neg q, \neg p \vee q$

$q \vee \neg q$

$\emptyset$

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$$p \lor r, p \lor \neg s, \neg p \lor s, \neg p \lor \neg r, s \lor \neg r, \neg s \lor r$$

$$(r \land \neg s) \lor p, (\neg r \land s) \lor \neg p, s \lor \neg r, \neg s \lor r$$

$$(r \land \neg s) \lor (\neg r \land s), s \lor \neg r, \neg s \lor r$$

$$s \lor r, \neg s \lor \neg r, s \lor \neg r, \neg s \lor r$$

$$(s \land \neg s) \lor r, (s \land \neg s) \lor \neg r$$

$$(s \land \neg s)$$

$$s, \neg s = \bot$$

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$$\exists x \exists y \forall z ((F(x,y) \rightarrow (F(y,z) \wedge F(z,z)))$$
$$\wedge ((F(x,y) \wedge G(x,y)) \rightarrow (G(x,z) \wedge G(z,z))))$$

$$F(x,y),$$
$$\neg F(y, f(x,y)) \vee \neg F(f(x,y), f(x,y)) \vee G(x,y),$$
$$\neg F(y, f(x,y)) \vee \neg F(f(x,y), f(x,y)) \vee \neg G(x, f(x,y)) \vee \neg G(f(x,y), f(x,y))$$

It took 25 lines of instantiation using the enumeration procedure described before, where nesting of functional symbols goes up to 25. Then, unit resolution can be repeatedly applied and a contradiction is found.

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- Rule for Eliminating Atomic Formulas: if we can obtain (by factoring) $\mathcal{S} = \{(\varphi \vee p), (\psi \vee \neg p), R\}$, where R is free of p, then $\mathcal{S} = \{(\varphi \vee \psi), R\}$ (1960).
- Splitting (Rule for Eliminating Atomic Formulas)*: if we can obtain (by factoring) $\mathcal{S} = \{(\varphi \vee p), (\psi \vee \neg p), R\}$, where R is free of p, then $\mathcal{S} = \{(\varphi \wedge R), (\psi \wedge R)\}$ (1962).
- One of the formulae is put on the tape and processed; if it is not satisfiable, then the other formula is tested for satisfiability.
- The formula

$$\exists x \exists y \forall z ((F(x,y) \rightarrow (F(y,z) \wedge F(z,z))) \wedge ((F(x,y) \wedge G(x,y)) \rightarrow (G(x,z) \wedge G(z,z)))$$

was proved under two minutes after 60 instantiation lines.

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▷ Resolution
John Alan Robinson
Argonne
1962-1963
... and then came light!
The Resolution Principle
Resolution, at last
Unification
Another Example
The upshot
Literal Unification
Occurs Check
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Resolution

John Alan Robinson

- "Theories of Meaning Implicit in the British Empiricists Locke, Berkeley and Hume" (Cambridge, 1952).
- "Causality, Probability and Testimony" (PhD, University of Oregon, University of Princeton, 1956).
- DuPont, operations research.

- In 1960, while holding a postdoc at the University of Pittsburgh, he accepts a tenure from Rice University, but decides to take a Summer research position at the Applied Mathematics Division of the Argonne National Laboratory (Chicago).
- His appointment at Argonne was to implement the Davis-Putnam procedure on an IBM 704.
- This work is described in a report (1961) which is in revised form in a paper from the Journal of the ACM [Rob63].

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In 1960, he implements the DP procedure in FORTRAN (together with George Robinson).

> I spent the summer [of 1961] happily... running this thing. Yes, it worked just like they [Davis and Putnam] said. However, the examples that one would like to try next blew it up, and we ran into combinatorial explosion of the instantiation, 'try all the instances, enumerate all the instances.' So I learnt that summer that was not exactly an elegant way to go.

In 1961, he implements Prawitz procedure in FORTRAN.

> The work at Argonne really was dominating my life, because I started looking at other ways to go in this first-order procedure stuff, and stumbled across the work of Prawitz, who was the one who reached some form of unification, calculating instances rather than trying them all.

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William F. Miller, the boss at the Applied Mathematics Division in Argonne, sets up a team for the theorem-proving research project:

- John Alan Robinson.
- George Robinson, who earlier had helped with the implementation of the DP procedure.
- Larry Wos, a mathematician at the Lab.
- During those two summer projects, Robinson rediscovered unification and the resolution rule, which was submitted in 1963, sitting on the desk of some reviewer for a year, and finally published in 1965.

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- On the unification procedure, Robinson says that he was "absolutely inspired by Prawitz". He uses a version similar to that of Herbrand's unification algorithm in his paper. Herbrand's procedure is definitional rather than the one devised by Prawitz, which is algorithmic.
- Prawitz and Robinson were not aware of this part of Herbrand's paper (although both cite the paper).
- The resolution rule had also appeared before:
  - it was first discovered in 1937, by Blake [Bla37];
  - then rediscovered by Quine (there called "consensus rule"), in 1955 [vOQ55];
  - and it was proposed for theorem proving in 1962 at Harvard University by Dunham and North [DN63].

The Resolution Principle

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> Traditionally, a single step in deduction has been required, for pragmatic and psychological reasons, to be simple enough, broadly speaking, to be apprehended as correct by a human being in a single intellectual act. No doubt this custom originated in the desire that each single step of a deduction should be indubitable, even though the deductions as whole may consist of a long chain of such steps.

Resolution was far more complex, as it "condones single inferences which are often beyond the ability of human to grasp (other than discursively)." Resolution is *machine-oriented*: it was shown to be complete and it was designed for efficiency. Thus, although indubitable, resolution wasn't meant for human reading.

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$$[\text{CUT}] \quad \frac{\begin{array}{ccc} \varphi & \Rightarrow & \psi \\ \psi & \Rightarrow & \chi \end{array}}{\begin{array}{ccc} \varphi & \Rightarrow & \chi \end{array}}$$

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$$[\text{CUT}]\quad \frac{\begin{array}{ccc}\varphi & \Rightarrow & \psi\\ \psi & \Rightarrow & \chi\end{array}}{\begin{array}{ccc}\varphi & \Rightarrow & \chi\end{array}}$$

$$[\text{RESOLUTION}]\quad \frac{\begin{array}{ccc}(\varphi & \vee & \psi)\\ (\neg\psi & \vee & \chi)\end{array}}{\begin{array}{ccc}(\varphi & \vee & \chi)\end{array}}$$

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$$[\text{CUT}] \quad \frac{\begin{array}{ccc}\varphi & \Rightarrow & \psi \\ \psi & \Rightarrow & \chi\end{array}}{\begin{array}{ccc}\varphi & \Rightarrow & \chi\end{array}}$$

$$[\text{RESOLUTION}] \quad \frac{\begin{array}{ccc}(\varphi & \vee & \psi) \\ (\neg\psi & \vee & \chi)\end{array}}{\begin{array}{ccc}(\varphi & \vee & \chi)\sigma\end{array}}$$

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$$[\text{RESOLUTION}] \quad \frac{\begin{array}{ccc}(\varphi & \vee & \psi) \\ (\neg\psi' & \vee & \chi)\end{array}}{(\varphi \quad \vee \quad \chi)\sigma}$$

where $\sigma = \text{unifier}(\psi, \psi')$

Example:

$$\begin{array}{lll} 1. & \neg Rich(x) \vee Happy(x) & \\ 2. & Rich(Bill) & \\ 3. & Happy(Bill) & [RES, 1, 2, Bill \backslash x] \end{array}$$

Another Example

1. $F(x, y)$
2. $\neg F(z, f_1(w, z)) \vee \neg F(f_1(w, z), f_1(w, z)) \vee G(w, z)$
3. $\neg F(t, f_1(u, t)) \vee \neg F(f_1(u, t), f_1(u, t)) \vee \neg G(u, f_1(u, t)) \vee \neg G(f_1(u, t), f_1(u, t))$
4. $\neg F(f_1(w, z), f_1(w, z)) \vee G(w, z)$
   $[Res, 1, 2, z \backslash x, f_1(w, z) \backslash y]$
5. $G(w, z)$
   $[Res, 1, 5, z \backslash x, f_1(w, z) \backslash y]$
6. $\neg F(f_1(w, z), f_1(w, z)) \vee \neg G(w, f_1(w, z)) \vee \neg G(f_1(x, y), f_1(x, y))$
   $[Res, 1, 3, z \backslash x, f_1(w, z) \backslash y]$
7. $\neg G(t, f_1(u, t)) \vee \neg G(f_1(u, t), f_1(u, t))$
   $[Res, 1, 6, t \backslash x, f_1(u, t) \backslash y]$
8. $\neg G(t, f_1(u, t)) \vee \neg G(f_1(u, t), f_1(u, t))$
   $[Res, 1, 7, t \backslash x, f_1(u, t) \backslash y]$
9. $\neg G(f_1(u, t), f_1(u, t))$
   $[Res, 8, 5, t \backslash w, f_1(u, t) \backslash z]$
10. $\square$
   $[Res, 9, 5, f_1(u, t) \backslash w, f_1(u, t) \backslash z]$

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- Argonne was then the world champion in theorem-proving!
- Refinements were mostly introduced by Robinson and Wos.
- From 1967 till 1970, Robinson publishes around 90 papers on theorem-proving.

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- Let φ be a literal. We denote by φ^i the symbol at the $i-th$ position of φ. If $i > |\varphi|$, then φ^i is undefined.
- Let φ and ψ be literals with $\varphi \neq \psi$. Let i be the least number such that both φ^i and ψ^i are defined and such that $\varphi^i \neq \psi^i$. The pair (φ^i, ψ^i) is called the *disagreement pair* of φ and ψ.

Example:

$$\begin{array}{rcl} \varphi & = & P(g_1(c), f_1(a, g_1(x), g_2(a, g_1(b)))) \\ \psi & = & P(g_1(c), f_1(a, g_1(x), g_2(f_2(x, y), z))) \end{array}$$

Disagreement pair: $(a, f_2(x, y))$

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- A substitution $[t\backslash x]$ satisfies the occurs check iff x does not occur in t.
- A disagreement pair $D = (x, t)$ satisfies the occurs check if the variable x does not occur in t.

Examples:

$$\begin{array}{rcl} \varphi & = & P(a, x, f(g(y))) \\ \psi & = & P(z, h(z, w), f(w)) \end{array}$$

$D = (a, z)$ satisfies the occurs check.

$$\begin{array}{rcl} \varphi & = & P(z, h(z, w), f(w)) \\ \psi & = & P(f(z), h(z, w), f(w)) \end{array}$$

$D = (z, f(z))$ does not satisfy the occurs check.

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Most General Unifier

Input: a pair of expressions χ_1 and χ_2.

Output: m.u.g. σ, if χ_1 and χ_2 are unifiable; "no", otherwise.

$\sigma \leftarrow \emptyset$; $W \leftarrow \{\chi_1, \chi_2\}$; $D \leftarrow$ is the disagreement pair of W;
while $|W| > 1$ and D satisfies the occurs check:
  select a variable x and a term t in D (x does not occur in t)
  $\sigma \leftarrow \sigma \circ \{t \backslash x\}$
  $W = \{\chi_1, \chi_2\} \leftarrow \{\chi_1[t \backslash x], \chi_2[t \backslash x]\}$
  $D \leftarrow$ disagreement pair of W
end_while
if $|W| = 1$
then return σ
else return "no"

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$\chi_1 = P(a, x, f(y))$ e $\chi_2 = P(u, v, w)$

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$\chi_1 = P(a, x, f(y))$ e $\chi_2 = P(u, v, w)$

$\sigma \leftarrow \emptyset$; $W \leftarrow \{P(a, x, f(y)), P(u, v, w)\}$; $D \leftarrow (a, u)$

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$\sigma \leftarrow \emptyset$; $W \leftarrow \{P(a, x, f(y)), P(u, v, w)\}$; $D \leftarrow (a, u)$

$|W| > 1$ and D satisfies the occurs check

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$\sigma \leftarrow \emptyset$; $W \leftarrow \{P(a, x, f(y)), P(u, v, w)\}$; $D \leftarrow (a, u)$

$|W| > 1$ and D satisfies the occurs check

$\sigma \leftarrow \emptyset \circ \{a \backslash u\} = \{a \backslash u\}$

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- $|W| > 1$ and D satisfies the occurs check
- $\sigma \leftarrow \emptyset \circ \{a \backslash u\} = \{a \backslash u\}$
- $W \leftarrow \{P(a, x, f(y))[a \backslash u], P(u, v, w)[a \backslash u]\}$
- $W = \{P(a, x, f(y)), P(a, v, w)\}$, $D \leftarrow (x, v)$

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$W = \{P(a, x, f(y)), P(a, v, w)\}$, $D \leftarrow (x, v)$

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$\sigma \leftarrow \{a \backslash u\} \circ \{v \backslash x\} = \{a \backslash u, v \backslash x\}$

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$W = \{P(a, x, f(y)), P(a, v, w)\}$, $D \leftarrow (x, v)$

$|W| > 1$ and D satisfies the occurs check

$\sigma \leftarrow \{a \backslash u\} \circ \{v \backslash x\} = \{a \backslash u, v \backslash x\}$

$W \leftarrow \{P(a, x, f(y))[v \backslash x], P(a, v, w)[v \backslash x]\}$

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$W \leftarrow \{P(a, x, f(y))[a \backslash u], P(u, v, w)[a \backslash u]\}$

$W = \{P(a, x, f(y)), P(a, v, w)\}$, $D \leftarrow (x, v)$

$|W| > 1$ and D satisfies the occurs check

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$W \leftarrow \{P(a, x, f(y))[v \backslash x], P(a, v, w)[v \backslash x]\}$

$W = \{P(a, v, f(y)), P(a, v, w)\}$, $D \leftarrow (f(y), w)$

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- $|W| > 1$ and D satisfies the occurs check
- $\sigma \leftarrow \emptyset \circ \{a \backslash u\} = \{a \backslash u\}$
- $W \leftarrow \{P(a, x, f(y))[a \backslash u], P(u, v, w)[a \backslash u]\}$
- $W = \{P(a, x, f(y)), P(a, v, w)\}$, $D \leftarrow (x, v)$
  - $|W| > 1$ and D satisfies the occurs check
  - $\sigma \leftarrow \{a \backslash u\} \circ \{v \backslash x\} = \{a \backslash u, v \backslash x\}$
  - $W \leftarrow \{P(a, x, f(y))[v \backslash x], P(a, v, w)[v \backslash x]\}$
  - $W = \{P(a, v, f(y)), P(a, v, w)\}$, $D \leftarrow (f(y), w)$
    - $|W| > 1$ and D satisfies the occurs check
    - $\sigma \leftarrow \{a \backslash u, v \backslash x\} \circ \{f(y) \backslash w\} = \{a \backslash u, v \backslash x, f(y) \backslash w\}$

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$\chi_1 = P(a, x, f(y))$ e $\chi_2 = P(u, v, w)$

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$W = \{P(a, v, f(y))\}$; D is undefined;

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$W \leftarrow \{P(a, v, f(y))[f(y) \backslash w], P(a, v, w)[f(y) \backslash w]\}$

$W = \{P(a, v, f(y))\}$; D is undefined;

$|W| = 1$, returns $\sigma = \{a \backslash u, v \backslash x, f(y) \backslash w\}$

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The method proposed by Robinson in [Rob65b].

- □ There are no axioms.
- □ There is only one inference rule:

$$[\text{RESOLUTION}] \quad \frac{\begin{array}{ccc}(\varphi & \vee & l) \\ (\psi & \vee & \neg l)\end{array}}{\begin{array}{ccc}(\varphi & \vee & \psi)\end{array}}$$

- □ Premises are called parent clauses (or the resolvends).
- □ The conclusion is called the resolvent.
- □ The literals l and $\neg l$ are known as complementary literals.
- □ The parent clauses are *resolved* on the the complementary literals, generating the resolvent.

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Let φ be a propositional formula and Γ_0 be the set of clauses resulting from the transformation of φ into CNF.

do set $\Gamma = \Gamma_0$

select $l, c_1 \in \Gamma, c_2 \in \Gamma$ such that $\left\{ \begin{array}{l} l \in c_1, \neg l \in c_2, \\ c_1 \text{ and } c_2 \text{ have} \\ \text{not been resolved} \end{array} \right\}$

compute the resolvent r

replace Γ_0 by $\Gamma_0 \cup \{r\}$

until $\Box \in \Gamma_0$ or $\Gamma_0 = \Gamma$

Note that the algorithm is non-deterministic.

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Let $\Gamma_i = \Gamma_0$
repeat
    Select c_1 e $c_2 \in \Gamma_i$ such that $l \in c_1, \neg l \in c_2$, where l is a literal and c_1 and c_2 have not been resolved
    Compute the resolvent r
    if not((r a tautology) or ($r \in \Gamma_i$)) **then**
        Do $\Gamma_{i+1} = \Gamma_i \cup \{r\}$
    end if
until $\square \in \Gamma_0$ or $\Gamma_{i+1} = \Gamma_i$

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There are two inference rules:

$$\frac{\begin{array}{ccc} D & \vee & L_1 \\ D' & \vee & \neg L_2 \end{array}}{(D \vee D')\sigma}$$

where σ is the most general unifier for L_1 and L_2.

And we also need factoring:

Let $C \stackrel{\text{def}}{=} L_1 \vee \ldots \vee L_n$ be a clause and σ the most general unifier for a subset of C. Then, $C\sigma$ is the *factor* of C and is added to the set of clauses.

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1. Let Γ_i be a set of clauses;
2. Choose clauses C_1 and C_2 with complementary literals in Γ_i;
3. Compute the resolvent R from C_1 and C_2, applying factoring;
4. If R is the empty clause, then stop and return "unsatisfiable";
5. Otherwise, $N_{i+1} \leftarrow N_i \cup \{R\}$;
6. If $N_{i+1} = N_i$, then stop and return "satisfiable";
7. Otherwise, go back to step 2.

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1. Resolution for first-order is strongly sound:

$$\text{if } \Gamma \vdash^{Res}_{\mathcal{L}_{fo}} \varphi, \text{ then } \Gamma \models_{\mathcal{L}_{fo}} \varphi$$

2. Resolution for first-order is refutationally complete:

$$\text{if } \Gamma \models_{\mathcal{L}_{fo}} \bot, \text{ then } \Gamma \vdash^{Res}_{\mathcal{L}_{fo}} \bot$$

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2. Resolution for first-order is refutationally complete:

$$\text{if } \Gamma \models_{\mathcal{L}_{fo}} \bot\text{, then } \Gamma \vdash^{Res}_{\mathcal{L}_{fo}} \bot$$

Binary resolution is also consequence complete:

Lemma 1 ([Lee67]). Let C be a set of propositional clauses. If a clause D is a consequence of C, then there is a clause D' which is derived by binary propositional resolution from C and D' subsumes D.

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Lemma 2. Let Γ be a set of clauses such that clauses $C, C' \in \Gamma$ and $C \models C'$. Γ is unsatisfiable if, and only if, $\Gamma \setminus \{C'\}$ is unsatisfiable.

Obs.: If $C \models C'$, we say that C *subsumes* C'.

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▷ Historical References

The following is a list of historical accounts, some of which are given by people working in the early years of Automated Reasoning:

[Dav83], [Lov84], [BL84], [Bib07], and [Dav01] .

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